Lecture 2

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1 Introduction to linear equations

Last lecture we were talking about the general mathematical concepts, like a concept of a number, a concept of a set, a concept of an operation. This lecture we will start studying concepts of linear algebra itself.

The first main thing which appears in linear algebra is a linear equation.

Definition 1.1. *The* **linear equation** *is the equation which can be transposed to the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

Example 1.2. *The equation*

$$0x_1 + 3x_2 + 4x_3 + 0x_4 = 6$$

is a linear equation, but the equation

$$4x_1x_2 + 3x_3 = 5$$

is not, since it contains the term $4x_1x_2$*, and so can not be transposed to the form (1).*

Here, we have 3 different types of numbers. First, a_i — they are called the coefficients of an equation; then, b — the constant of an equation; both a_i's and b are given numbers. Also, we have x_i's — variables, which are not given, but should be determined from this equation. Our main goal is to solve this equation. What does it mean "to solve" the equation???

Definition 1.3. *A* **solution** *of a linear equation (1) is a n-tuple of numbers* $(k_1, k_2, \ldots, k_n)$ *such that*

$$a_1k_1 + \cdots + a_nk_n = b$$

Example 1.4. *Let's consider the equation*

$$0x_1 + 3x_2 + 4x_3 = 7$$

Then $(1, 2, 3)$ *is not a solution for this equation since* $0 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 = 18 \neq 7$*, and* $(1, 1, 1)$ *is a solution, since* $0 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 = 7$*. Moreover,* $\forall k \in \mathbb{R}$*,* $(k, 1, 1)$ *is a solution for this equation. So, we see, that this equation has infinitely many solutions.*

"To solve" the equation is to find all its solutions.

2 The easiest equation

Now we'll proceed to the easiest type of linear equation — linear equation with one variable. It is an equation of the following form:

$$ax = b \tag{2}$$

Let's analyze it. We have 3 cases for this equation.

- **Case 1.** $a \neq 0$, $b \neq 0$. In this case the equation has only one solution, which can be obtained by the following formula
$$x = \frac{b}{a}$$
This is a solution. To check it let's substitute this value to the equation; we will have: $a \cdot \frac{b}{a} = b$. Moreover this is a unique solution: multiplying $ax = b$ by $\frac{1}{a}$ we get $x = b \cdot \frac{1}{a} = \frac{b}{a}$.

- **Case 2.** $a = 0$, $b = 0$. In this case left hand side of the equation is equal to 0 for any x, and right hand side is 0. So, any number is a solution of this equation. So, in this case the equation has infinitely many solutions.

- **Case 3.** $a = 0$, $b \neq 0$. In this case left hand side of the equation is equal to 0 for any number x, and right hand side is not equal to 0. So, in this case the equation has no solutions.

So, we saw 3 different cases — the equation could have 0, 1 or infinitely many solutions. Later we will see that this is the general case.

3 General linear equation

Now we will proceed to the general form of a linear equation. As we've already seen, it can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b. \tag{3}$$

We'll get the general solution for this equation.

First we'll need some definitions.

Definition 3.1. *The variable* x_j *in the equation (3) is called* **the leading variable** *if the following 2 conditions are satisfied:*

1. $\forall \quad i < j \quad a_i = 0$

2. $a_j \neq 0$.

In other words, the leading variable is the first variable which has a nonzero coefficient.

Example 3.2. *For the equation* $0x_1 + 0x_2 + 0x_3 + 2x_4 + 3x_5 = 7$ *the leading variable is* x_4.

Note that an equation may not have any leading variables, if all a_i's are equal to 0.

Now let's try to solve the general linear equation. Suppose it has a leading variable x_j. Then we will give *any* values to all variables other than x_j, say $x_1 = k_1, x_2 = k_2, \ldots x_{j-1} = k_{j-1}, x_{j+1} = k_{j+1}, \ldots, x_n = k_n$ (we're skipping x_j!!!). After that the value of x_j can be obtained by the following formula:

$$x_j = \frac{b - a_1k_1 - \cdots - a_{j-1}k_{j-1} - a_{j+1}k_{j+1} - \cdots - a_nk_n}{a_j}. \tag{4}$$

This can be easily obtained by isolating the term a_jx_j in the left hand side of the equation (3), and dividing by a_j after that. Now, the general solution of the equation (3) is

$$x_1 = k_1; \ldots; x_{j-1} = k_{j-1};$$

$$x_j = \frac{b - a_1k_1 - \cdots - a_{j-1}k_{j-1} - a_{j+1}k_{j+1} - \cdots - a_nk_n}{a_j};$$

$$x_{j+1} = k_{j+1}; \ldots; x_n = k_n.$$

Example 3.3. *Let's solve the equation*

$$0x_1 + 2x_2 - 6x_3 = 10$$

The leading variable for this equation is x_2. *So, let* $x_1 = k_1$, *and* $x_3 = k_3$. *Then,*

$$x_2 = \frac{10 - 0k_1 + 6k_3}{2} = 5 + 3k_3.$$

So, the solution is

$$x_1 = k_1; x_2 = 5 + 3k_3; x_3 = k_3,$$

where k_1 *and* k_3 *can be any real numbers. Now using this general form of the solution we can get as many solutions as we want. For example, choosing* $k_1 = 5$ *and* $k_3 = 10$ *we'll get the solution* $(5, 35, 10)$.

Now, if there are no leading variables for this equation, than we have 2 different cases.

Case 1. $b = 0$. Then the equation has the form

$$0x_1 + \cdots + 0x_n = 0,$$

and any n-tuple $(k_1, \ldots, k_n)$ is a solution.

Case 2. $b \neq 0$. Then the equation has the form

$$0x_1 + \cdots + 0x_n = b,$$

and the left hand side of an equation is always 0, but right hand side is not. So the equation has no solution.

As we can see, we got the same result, as in the easiest case: **The equation can have 0, 1 or infinitely many solutions.**